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ORIGINAL PAPER

Rational groups and integer-valued characters of Thompson group Th

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Abstract According to the main result of W. Feit and G. M. Seitz (Illinois J Math 33(1):103–131, 1988), the Thompson group Th is non-rational or unmatured group (S. Fujita in Bull Chem Soc Jpn 71:2071–2080, 1998). Using the concept of markaracter tables proposed by S. Fujita (Bull Chem Soc Jpn 71:1587–1596, 1998), we are able to obtain tables of integer-valued characters for finite unmatured groups. In this paper, the integer-valued character for Thompson group is successfully derived for the first time.

Keywords Rational group · Integer-valued characters · Matured groups · Dominant classes · Markaracter · Thompson group

1 Introduction

Shinsaku Fujita suggested a new concept called the markaracter table, which enables us to discuss marks and characters for a finite group on a common basis, and then introduced tables of integer-valued characters [5,10], which are acquired for such groups. Fujita's theory was further developed and utilized for a variety of enumeration problems of chemical species eventually [4–6,8–12]. A dominant class is defined as a disjoint union of conjugacy classes corresponding the same cyclic subgroups, which is selected as a representative of conjugate cyclic subgroups [5–7,10].

The Thompson group of order 90745943887872000 is an unmatured group according to the main result of W. Feit and G. M. Seitz in [3]. The motivation for this study is outlined in [2,4], and [5], and the reader is encouraged to consult the papers and

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[1,13,15], and [14] for background material as well as basic computational techniques. This paper is organized as follows: In Sect. 2, we introduce some necessary concepts, such as the maturity, $\mathbb{Q}$-group and $\mathbb{Q}$-conjugacy character of a finite group. In Sect. 3, we provide all the dominant classes and $\mathbb{Q}$-conjugacy characters for the the Thompson group Th.

2 Preliminaries

Throughout this paper we adopt the same notations as in [4,5]. We will use the ATLAS notations [1] for conjugacy classes. Thus, nx, n is an integer and $x = a, b, c, \ldots$ denote conjugacy classes of G of elements of order n.

Before stating discussion, we will mention some well-known results about $\mathbb{Q}$-conjugation. An alternative characterization of $\mathbb{Q}$-conjugation is the following concepts which can be found [5–7,10,11].

A *dominant class* is defined as a disjoint union of conjugacy classes that corresponds to the same cyclic subgroup, which is selected as a representative of conjugate cyclic subgroups. Let G be a finite group and $h_1, h_2 \in G$. We say h_1 and h_2 are $\mathbb{Q}$-conjugate if $t \in G$ exists such that $t^{-1}\langle h_1\rangle t = \langle h_2\rangle$ which is an equivalence relation on group G and generates equivalence classes that are called dominant classes. The group G is partitioned in to dominant classes as follows: $G = K_1 + K_2 + \cdots + K_s$ in which K_i corresponding to the cyclic (dominant) subgroup G_i selected from a non-redundant set of cyclic subgroups of G denoted by $SCSG$.

Let C be a $m \times m$ matrix of the character table for an arbitrary finite group G. Then, C is transformed into a more concise form called the $\mathbb{Q}$-conjugacy character table denoted by $C_G^{\mathbb{Q}}$ containing integer-valued characters. By Theorem 4 in [5], the dimension of a $\mathbb{Q}$-conjugacy character table, $C_G^{\mathbb{Q}}$ is equal to its corresponding markaracter table, i.e., $C_G^{\mathbb{Q}}$ is an $n \times n$-matrix where n is the number of dominant classes or equivalently the number of $SCSG$. If $m = n$, then $C = C^{\mathbb{Q}}$ i.e. G is a *maturated* group. Otherwise, $n < m$ (is called *unmaturated* group) for each $G_i \in SCGG$ (the corresponding dominant class K_i) set $t_i = m(G_i)/\varphi(|G_i|)$ where $m(G_i) = |N_G(G_i)|/|C_G(G_i)|$ (called the maturity discriminant), φ is the Euler function. If $t_i = 1$ then, K_i is exactly a conjugacy class so there is no reduction in row and column of C but if $t_i > 1$ then K_i is a union of t_i-conjugacy classes of G (i.e. reduction in column) therefore the sum of t_i rows of irreducible characters via the same degree in C (reduction in rows) gives us a reducible character which is called the $\mathbb{Q}$-conjugacy character.

Now, we recall some concepts of rational group theory. Let G be a finite group and χ be a complex character of G. Let $\mathbb{Q}(\chi)$ denote the subfield of the complex numbers $\mathbb{C}$ generated by $\mathbb{Q}$ and all the values $\chi(x)$, $x \in G$, where $\mathbb{Q}$ denotes the field of rational numbers. By definition, χ is called rational if $\mathbb{Q}(\chi) = \mathbb{Q}$. A finite group G is called a rational group or a $\mathbb{Q}$-group, if all irreducible complex characters of G are rational. For example, the symmetric group S_n and the Weyl groups of the classical complex Lie algebras are rational groups (for more details see [1]). A comprehensive description of rational groups can be found in [16].

Table 1 The integer-valued character table of Thompson group Th where $A_{12} = 12a \cup 12b$ is an unmatured dominat class

$C_{Th}^{\mathbb{Q}}$	1*a*	2*a*	3*a*	3*b*	3*c*	4*a*	4*b*	5*a*	6*a*	6*b*	6*c*	7*a*	8*a*	8*b*	9*a*	9*b*	9*c*	10*a*	A_{12}
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	248	−8	14	5	−4	8	0	−2	4	−2	1	3	0	0	5	−4	2	2	2
χ_3	4,123	27	64	−8	1	27	−5	−2	9	0	0	7	3	−1	−8	1	4	2	0
χ_4	54,000	240	−54	54	0	16	0	0	0	−6	6	2	0	0	0	0	0	0	−2
χ_5	30,628	−92	91	10	10	36	4	3	10	−5	−2	3	−4	0	10	10	1	3	3
χ_6	30,875	155	104	14	5	27	−5	0	5	8	2	5	3	−1	14	5	2	0	0
χ_7	61,256	72	182	20	20	56	0	6	12	6	0	6	0	0	−7	−7	2	2	2
χ_8	171,990	−42	0	−54	54	−42	22	−10	6	0	−6	0	6	−2	0	0	0	−2	0
χ_9	147,250	50	181	−8	−35	34	10	0	5	5	−4	5	2	−2	19	−8	1	0	1
χ_{10}	1,535,274	810	0	0	0	−54	−6	24	0	0	0	6	−6	−6	0	0	0	0	0
χ_{11}	1,558,494	−546	−378	−108	0	126	−18	−6	0	6	12	0	−2	6	0	0	0	−6	6
χ_{12}	957,125	−315	650	−52	−25	133	5	0	15	−6	0	8	−3	1	−25	2	2	0	−2
χ_{13}	3,414,528	−1,536	0	−108	108	0	0	28	−12	0	12	12	0	0	0	0	0	4	0
χ_{14}	2,450,240	832	260	71	44	64	0	−10	4	4	−5	−5	0	0	17	−10	−1	2	4
χ_{15}	2,572,752	−1,072	624	111	84	48	0	2	−4	−16	−1	7	0	0	30	3	3	−2	0
χ_{16}	3,376,737	609	819	9	9	161	1	−13	9	3	−3	0	1	1	9	9	0	−1	−1
χ_{17}	8,192,000	0	128	−16	−160	0	0	0	0	0	0	−16	0	0	−16	2	8	0	0
χ_{18}	4,123,000	120	118	19	−80	8	0	0	0	6	3	−7	0	0	19	1	4	0	2
χ_{19}	4,881,384	1,512	729	0	0	72	24	9	0	9	0	4	8	0	0	0	0	−3	−3

Table 1 continued

$C^{\mathbb{Q}}_{Th}$	1*a*	2*a*	3*a*	3*b*	3*c*	4*a*	4*b*	5*a*	6*a*	6*b*	6*c*	7*a*	8*a*	8*b*	9*a*	9*b*	9*c*	10*a*	A_{12}
χ_{20}	4,936,750	−210	637	−38	−65	126	−10	0	15	−3	6	0	−2	2	16	−11	−2	0	−3
χ_{21}	13,338,000	−2,160	−702	216	0	112	0	0	0	18	0	4	0	0	0	0	0	0	−2
χ_{22}	13,392,000	−1,920	−756	270	0	128	0	0	0	12	6	6	0	0	0	0	0	0	−4
χ_{23}	10,822,875	−805	924	141	−75	91	−5	0	5	−4	5	0	3	−1	−21	6	−3	0	4
χ_{24}	11,577,384	552	351	135	0	−120	24	9	0	15	3	7	−8	0	0	0	0	−3	3
χ_{25}	16,539,120	2,544	0	297	−54	48	16	−5	−6	0	−3	3	0	0	0	0	0	−1	0
χ_{26}	18,154,500	1,540	−273	213	−30	−28	20	0	10	−17	1	0	−4	0	−3	−3	−3	0	−1
χ_{27}	42,653,520	336	0	−270	−216	−336	0	20	24	0	−6	0	0	0	0	0	0	−4	0
χ_{28}	28,861,000	840	1,078	−110	160	56	0	0	0	6	−6	0	0	0	−29	−2	−2	0	2
χ_{29}	30,507,008	0	896	−184	32	0	0	8	0	0	0	0	0	0	32	5	−4	0	0
χ_{30}	40,199,250	3,410	−78	3	165	−62	10	0	5	2	−1	−7	−6	2	3	3	3	0	−2
χ_{31}	44,330,496	3,584	168	6	−156	0	0	−4	−4	8	2	0	0	0	6	6	−3	4	0
χ_{32}	51,684,750	2,190	0	108	135	−162	−10	0	15	0	12	−9	6	−2	0	0	0	0	0
χ_{33}	72,925,515	−2,997	0	0	0	27	51	15	0	0	0	−9	3	3	0	0	0	3	0
χ_{34}	76,271,625	−2,295	729	0	0	153	−15	0	0	9	0	−11	−7	−3	0	0	0	0	−3
χ_{35}	77,376,000	2,560	1,560	−60	−60	0	0	0	−20	−8	4	2	0	0	−6	−6	3	0	0
χ_{36}	81,153,009	−783	−729	0	0	225	9	9	0	−9	0	−7	1	−3	0	0	0	−3	3
χ_{37}	91,171,899	315	0	243	0	−21	−45	24	0	0	−9	0	3	3	0	0	0	0	0
χ_{38}	111,321,000	3,240	−1,728	−216	0	216	0	0	0	0	0	7	0	0	0	0	0	0	0
χ_{39}	190,373,976	−3,240	0	0	0	−216	0	−24	0	0	0	9	0	0	0	0	0	0	0

Table 2 The integer-valued character table of Thompson group Th where $A_n = na \cup nb$ for $n = 15, 24, 30, 31, 39$ $A_k = kb \cup kc$ for $k = 27, 36$ and $B_{24} = 24c \cup 24d$ are unmatured dominat class

$C_{Th}^{\mathbb{Q}}$	12*c*	12*d*	13*a*	14*a*	A_{15}	18*a*	18*b*	19*a*	20*a*	21*a*	A_{24}	B_{24}	27*a*	A_{27}	28*a*	A_{30}	A_{31}	36*a*	A_{36}	A_{39}
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	−1	0	1	−1	1	1	−2	1	0	0	0	0	2	−1	1	−1	0	−1	−1	1
χ_3	0	1	2	−1	1	0	0	0	0	1	0	−1	−2	1	−1	−1	0	0	0	−1
χ_4	−2	0	−2	2	0	0	0	2	0	2	0	0	0	0	2	0	−2	4	−2	−2
χ_5	0	−2	0	−1	0	−2	1	0	−1	0	−1	0	1	1	1	0	0	0	0	0
χ_6	0	1	0	1	0	2	2	0	0	−1	0	−1	2	−1	−1	0	−1	0	0	0
χ_7	2	0	0	2	0	−3	0	0	0	0	0	0	−1	−1	0	2	0	−1	−1	0
χ_8	−6	−2	0	0	−1	0	0	2	2	0	0	−2	0	0	0	1	2	0	0	0
χ_9	−2	1	−1	1	0	−1	−1	0	0	−1	−1	1	1	1	−1	0	0	1	1	−1
χ_{10}	0	0	0	−2	0	0	0	−2	4	0	0	0	0	0	2	0	−1	0	0	0
χ_{11}	0	0	2	0	0	0	0	0	2	0	−2	0	0	0	0	0	0	0	0	−1
χ_{12}	−2	−1	0	0	0	3	0	0	0	−1	0	1	−1	−1	0	0	0	1	1	0
χ_{13}	0	0	0	4	−2	0	0	0	0	0	0	0	0	0	0	−2	2	0	0	0
χ_{14}	1	0	0	−1	−1	1	1	0	0	1	0	0	−1	−1	1	−1	0	1	1	0
χ_{15}	3	0	0	−1	−1	2	−1	0	0	1	0	0	0	0	−1	1	0	0	0	0
χ_{16}	−1	1	0	0	−1	−3	0	0	1	0	1	1	0	0	0	−1	0	−1	−1	0
χ_{17}	0	0	−2	0	0	0	0	−2	0	2	0	0	2	−1	0	0	2	0	0	−2
χ_{18}	−1	0	−2	1	0	3	0	0	0	−1	0	0	−2	1	1	0	0	−1	−1	1
χ_{19}	0	0	1	0	0	0	0	−1	−1	1	−1	0	0	0	2	0	0	0	0	1
χ_{20}	0	−1	0	0	0	0	0	−1	0	0	1	−1	1	1	0	0	0	0	0	0

Table 2 continued

$C_{Th}^{\mathbb{Q}}$	$12c$	$12d$	$13a$	$14a$	A_{15}	$18a$	$18b$	$19a$	$20a$	$21a$	A_{24}	B_{24}	$27a$	A_{27}	$28a$	A_{30}	A_{31}	$36a$	A_{36}	A_{39}
χ_{21}	4	0	0	−4	0	0	0	0	0	−2	0	0	0	0	0	0	2	4	−2	0
χ_{22}	2	0	−2	−2	0	0	0	2	0	0	0	0	0	0	2	0	0	−4	2	−2
χ_{23}	1	1	−2	0	0	−1	−1	0	0	0	0	−1	0	0	0	0	0	1	1	1
χ_{24}	−3	0	0	−1	0	0	0	0	−1	1	1	0	0	0	−1	0	0	0	0	0
χ_{25}	3	−2	0	3	1	0	0	0	1	0	0	0	0	0	−1	−1	0	0	0	0
χ_{26}	−1	2	0	0	0	1	1	0	0	0	−1	0	0	0	0	0	1	−1	−1	0
χ_{27}	6	0	0	0	−1	0	0	2	0	0	0	0	0	0	0	−1	0	0	0	0
χ_{28}	2	0	−1	0	0	3	0	0	0	0	0	0	1	1	0	0	0	−1	−1	−1
χ_{29}	0	0	−1	0	2	0	0	0	0	0	0	0	−1	−1	0	0	1	0	0	−1
χ_{30}	1	1	0	1	0	−1	−1	0	0	−1	0	−1	0	0	1	0	0	1	1	0
χ_{31}	0	0	2	0	−1	2	−1	0	0	0	0	0	0	0	0	1	0	0	0	−1
χ_{32}	0	−1	0	−1	0	0	0	0	0	0	0	1	0	0	−1	0	0	0	0	0
χ_{33}	0	0	0	−1	0	0	0	0	1	0	0	0	0	0	−1	0	−1	0	0	0
χ_{34}	0	0	1	1	0	0	0	1	0	1	−1	0	0	0	−1	0	0	0	0	1
χ_{35}	0	0	0	−2	0	−2	1	1	0	−1	0	0	0	0	0	0	0	0	0	0
χ_{36}	0	0	2	1	0	0	0	0	−1	−1	1	0	0	0	1	0	0	0	0	−1
χ_{37}	−3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{38}	0	0	−2	−1	0	0	0	0	0	1	0	0	0	0	−1	0	0	0	0	1
χ_{39}	0	0	0	1	0	0	0	−1	0	0	0	0	0	0	1	0	0	0	0	0

Theorem 2.1 [16] *A group G is a $\mathbb{Q}$-group if and only if for every $x \in G$ of order n the elements x and x^m are conjugacy in G, whenever $(m, n) = 1$. Equivalency, for each $x \in G$ we must have $\frac{N_G(\langle x \rangle)}{C_G(\langle x \rangle)} \simeq Aut(\langle x \rangle)$.*

The following depth Theorem due to Fiet and Siet [3].

Theorem 2.2 *Let G be a noncyclic simple group. Then G is a $\mathbb{Q}$-group if and only if $G \simeq Sp_6(2)$ or $O_8^+(2)'$.*

By Definition $\mathbb{Q}$-conjugacy class and Theorems 2.1 and 2.2, every $\mathbb{Q}$-group is matured.

3 Results and discussions

According to the Theorem 2.2, the Thompson group Th is an unmatured group. Now we are equipped to compute all the dominant classes and $\mathbb{Q}$-conjugacy characters for the above group, using a GAP program [13].[1]

Theorem 3.1 *The Thompson group Th has thirty nine dominant classes. Moreover, the unmaturated dominant classes of Th have orders 12, 15, 24, 24, 27, 30, 31, 36 and 39 with the corresponding maturities 2, 2, 2, 2, 2, 2, 2 and 2, respectively.*

Proof The dimension of a $\mathbb{Q}$-conjugacy character table, $C_{Th}^{\mathbb{Q}}$ is equal to its corresponding markaracter table for Th. To find the number of dominant classes, at first, we calculate the table of marks for Th [14,15] via GAP system, see GAP programs in [13] for more details.Hence, the markaracter table for Th corresponding to nine non-conjugate cyclic subgroups(i.e., $G_i \in SCS_{Th}$) of orders 12, 15, 24, 24, 27, 30, 31, 36 and 39.

Therefore, by using the above table, the character table of Th and Definition dominant class, since $|SCS_{Th}| = 9$, the dominant classes of Th are $A_{12} = 12a \cup 112b$, $A_n = na \cup nb$ for $n = 15, 24, 30, 31, 39$, $A_k = kb \cup kc$ for $k = 27, 36$ and $B_{24} = 24c \cup 24d$ with maturity (i.e., $t = \varphi(n)/m(H)$) 2, 2, 2, 2, 2, 2, 2, 2 and 2 respectively. □

The Thompson group Th has nine unmatured $\mathbb{Q}$-conjugacy characters. Furthermore Th has nine unmatured $\mathbb{Q}$-conjugacy characters χ_4, χ_8, χ_{10}, χ_{11}, χ_{13}, χ_{17}, χ_{21}, χ_{22} and χ_{27} which are the sum of two irreducible characters. Therefore, there are nine column-reductions (similarly nine row-reductions) in the character table of Th [4,5]. We provide all $\mathbb{Q}$-conjugacy characters of Th in Tables 1 and 2.

[1] which is available freely from: http://www.gap-system.org.

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